Prealgebra Textbook

Second Edition

Chapter 3

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Contents

Chapter 3

The Fundamentals of Algebra

As his name portends, Abu Jafr Muhammad ibn Musa al-Khwarizmi was one of the greatest Arab mathematicians of his time. While living in Baghdad during the ninth century AD he became the Chief Librarian at the *House of Wisdom*, a library and major center of intellectual study. In the year 820AD, al-Khwarizmi wrote *Al-Kitab al-mukhtasar ti Hisab al-jabr w'al-muqabala*, translated to, *The Compendious Book on Calculation by Restoration and Reduction*, the first book to generalize solving equations using the principles of equality. In fact, the word algebra itself comes from *al-jabr*, meaning reunion or completion.

Many earlier cultures had employed what we might call algebra in the service of business, land management, inheritance, and trade. The Bablyonians were solving quadratic equations in 2000BC, but only specific equations, with specific numbers. Hindus on the Indian continent were also developing algebra along side their invention of the symbol for zero 0. But like al-Khwarizmi and the Moslem Arabs of the first millenium, to write equations these early cultures did not use symbols like x or y, or even equal signs = that we use today. al-Khwarizmi wrote absolutely everything in words! 42 would be forty-two!

Early algebra written all with words is called rhetorical algebra, and a thousand years ago, each mathematician had their own way of expressing it. Algebra was a language with many different dialects, and communicating it from one mathematician to another was difficult as they had to explain themselves with words. It wasn't until well after the Gutenberg printing press was invented in 1436 and print became standardized, that Rene Descartes, a Frenchman began to develop a modern symbolic algebra.

In this section we'll learn how to manipulate symbols in order to al-muqabalah (combine like terms) and al-jabr (restore and balance equations). But we'll use modern notation to make it easier!

3.1 Mathematical Expressions

Recall the definition of a *variable* presented in Section 1.6.

Variable. A variable is a symbol (usually a letter) that stands for a value that may vary.

Let's add the definition of a *mathematical expression.*

Mathematical Expression. When we combine numbers and variables in a valid way, using operations such as addition, subtraction, multiplication, division, exponentiation, and other operations and functions as yet unlearned, the resulting combination of mathematical symbols is called a *mathematical expression.*

Thus,

$$2a, \qquad x+5, \qquad \text{and} \qquad y^2,$$

being formed by a combination of numbers, variables, and mathematical operators, are valid mathematical expressions.

A mathematical expression must be *well-formed*. For example,

$$2 + \div 5x$$

is *not a valid expression* because there is no term following the plus sign (it is not valid to write $+\div$ with nothing between these operators). Similarly,

$$2 + 3(2$$

is not well-formed because parentheses are not balanced.

Translating Words into Mathematical Expressions

In this section we turn our attention to translating word phrases into mathematical expressions. We begin with phrases that translate into *sums*. There is a wide variety of word phrases that translate into sums. Some common examples are given in Table 3.1(a), though the list is far from complete. In like manner, a number of phrases that translate into differences are shown in Table 3.1(b).

Let's look at some examples, some of which translate into expressions involving sums, and some of which translate into expressions involving differences.

You Try It!

Translate the following phrases into mathematical expressions: (a) "13 more than x", and (b) "12 fewer than y".

EXAMPLE 1. Translate the following phrases into mathematical expressions: (a) "12 larger than x," (b) "11 less than y," and (c) "r decreased by 9."

Solution. Here are the translations.

a) "12 larger than x" becomes $x + 12$.

b) "11 less than y" becomes $y - 11$.

c) "r decreased by 9" becomes $r - 9$.

Answers: (a) $x + 13$ and (b) $y - 12$

Phrase	Translates to:
sum of x and 12	$x + 12$
4 greater than b	$b + 4$
6 more than y	$y + 6$
44 plus r	$44 + r$
3 larger than z	$z + 3$

(a) Phrases that are sums.

Phrase	Translates to:
difference of x and 12	$x - 12$
4 less than b	$b - 4$
7 subtracted from y	$y - 7$
44 minus r	$44 - r$
3 smaller than z	$z - 3$

(b) Phrases that are differences.

Table 3.1: Translating words into symbols.

You Try It!

EXAMPLE 2. Let W represent the width of the rectangle. The length of a rectangle is 4 feet longer than its width. Express the length of the rectangle in terms of its width W.

The width of a rectangle is 5 inches shorter than its length L. Express the width of the rectangle in terms of its length L.

Solution. We know that the width of the rectangle is W. Because the length of the rectangle is 4 feet longer that the width, we must add 4 to the width to find the length.

Length	is	4	more than	the width
Length	$=$	4	$+$	W

Thus, the length of the rectangle, in terms of its width W, is $4 + W$.

Answer: $L - 5$

You Try It!

EXAMPLE 3. A string measures 15 inches is cut into two pieces. Let x represent the length of one of the resulting pieces. Express the length of the second piece in terms of the length x of the first piece.

A string is cut into two pieces, the first of which measures 12 inches. Express the total length of the string as a function of x, where x represents the length of the second piece of string.

Solution. The string has original length 15 inches. It is cut into two pieces and the first piece has length x. To find the length of the second piece, we must subtract the length of the first piece from the total length.

Length of the second piece	is	Total length	minus	the length of first piece
Length of the second piece	$=$	15	$-$	x

Thus, the length of the second piece, in terms of the length x of the first piece, is $15 - x$.

Answer: $12 + x$

□

There is also a wide variety of phrases that translate into products. Some examples are shown in Table 3.2(a), though again the list is far from complete. In like manner, a number of phrases translate into quotients, as shown in Table 3.2(b).

Phrase	Translates to:
product of x and 12	$12x$
4 times b	$4b$
twice r	$2r$

(a) Phrases that are products.

Phrase	Translates to:
quotient of x and 12	$x/12$
4 divided by b	$4/b$
the ratio of 44 to r	$44/r$

(b) Phrases that are differences.

Table 3.2: Translating words into symbols.

Let's look at some examples, some of which translate into expressions involving products, and some of which translate into expressions involving quotients.

You Try It!

Translate into mathematical symbols: (a) "the product of 5 and x" and (b) "12 divided by y."

EXAMPLE 4. Translate the following phrases into mathematical expressions: (a) "11 times x," (b) "quotient of y and 4," and (c) "twice a."

Solution. Here are the translations.

a) "11 times x" becomes $11x$.

b) "quotient of y and 4" becomes $y/4$, or equivalently, $\dfrac{y}{4}$.

c) "twice a" becomes $2a$.

Answer: (a) $5x$ and (b) $12/y$.

□

You Try It!

A carpenter cuts a board of unknown length L into three equal pieces. Express the length of each piece in terms of L.

EXAMPLE 5. A plumber has a pipe of unknown length x. He cuts it into 4 equal pieces. Find the length of each piece in terms of the unknown length x.

Solution. The total length is unknown and equal to x. The plumber divides it into 4 equal pieces. To find the length of each pieces, we must divide the total length by 4.

Length of each piece	is	Total length	divided by	4
Length of each piece	$=$	x	$\div$	4

Thus, the length of each piece, in terms of the unknown length x, is $x/4$, or equivalently, $\dfrac{x}{4}$.

Answer: $L/3$.

□

You Try It!

EXAMPLE 6. Mary invests A dollars in a savings account paying 2% interest per year. She invests five times this amount in a certificate of deposit paying 5% per year. How much does she invest in the certificate of deposit, in terms of the amount A in the savings account?

David invest K dollars in as savings account paying 3% per year. He invests half this amount in a mutual fund paying 4% per year. Express the amount invested in the mutual fund in terms of K, the amount invested in the savings account.

Solution. The amount in the savings account is A dollars. She invests five times this amount in a certificate of deposit.

Amount in CD	is	5	times	Amount in savings
Amount in CD	$=$	5	$\cdot$	A

Thus, the amount invested in the certificate of deposit, in terms of the amount A in the savings account, is $5A$.

Answer: $\frac{1}{2}K$

□

Combinations

Some phrases require combinations of the mathematical operations employed in previous examples.

You Try It!

EXAMPLE 7. Let the first number equal x. The second number is 3 more than twice the first number. Express the second number in terms of the first number x.

A second number is 4 less than 3 times a first number. Express the second number in terms of the first number y.

Solution. The first number is x. The second number is 3 more than twice the first number.

Second number	is	3	more than	twice the first number
Second number	$=$	3	$+$	$2x$

Answer: $3y - 4$

Therefore, the second number, in terms of the first number x, is $3 + 2x$. □

You Try It!

The width of a rectangle is W. The length is 7 inches longer than twice the width. Express the length of the rectangle in terms of its length L.

EXAMPLE 8. The length of a rectangle is L. The width is 15 feet less than 3 times the length. What is the width of the rectangle in terms of the length L?

Solution. The length of the rectangle is L. The width is 15 feet less than 3 times the length.

Width	is	3 times the length	less	15
Width	$=$	$3L$	$-$	15

Answer: $2W + 7$

Therefore, the width, in terms of the length L, is $3L - 15$. □

Exercises

In Exercises 1-20, translate the phrase into a mathematical expression involving the given variable.

1. "8 times the width n "
2. "2 times the length z "
3. "6 times the sum of the number n and 3"
4. "10 times the sum of the number n and 8"
5. "the demand b quadrupled"
6. "the supply y quadrupled"
7. "the speed y decreased by 33"
8. "the speed u decreased by 30"
9. "10 times the width n "
10. "10 times the length z "
11. "9 times the sum of the number z and 2"
12. "14 times the sum of the number n and 10"
13. "the supply y doubled"
14. "the demand n quadrupled"
15. "13 more than 15 times the number p "
16. "14 less than 5 times the number y "
17. "4 less than 11 times the number x "
18. "13 less than 5 times the number p "
19. "the speed u decreased by 10"
20. "the speed w increased by 32"

---

21. **Representing Numbers.** Suppose n represents a whole number.
    i) What does $n+1$ represent?
    ii) What does $n+2$ represent?
    iii) What does $n-1$ represent?
22. Suppose $2n$ represents an even whole number. How could we represent the next even number after $2n$?
23. Suppose $2n+1$ represents an odd whole number. How could we represent the next odd number after $2n+1$?
24. There are b bags of mulch produced each month. How many bags of mulch are produced each year?
25. Steve sells twice as many products as Mike. Choose a variable and write an expression for each man's sales.
26. Find a mathematical expression to represent the values.
    i) How many quarters are in d dollars?
    ii) How many minutes are in h hours?
    iii) How many hours are in d days?
    iv) How many days are in y years?
    v) How many months are in y years?
    vi) How many inches are in f feet?
    vii) How many feet are in y yards?

❧ ❧ ❧ Answers ☙ ☙ ☙

1. $8n$

3. $6(n+3)$

5. $4b$

7. $y-33$

9. $10n$

11. $9(z+2)$

13. $2y$

15. $15p+13$

17. $11x-4$

19. $u-10$

21.
i) $n+1$ represents the next whole number after n.

ii) $n+2$ represents the next whole number after $n+1$, or, two whole numbers after n.

iii) $n-1$ represents the whole number before n.

23. $2n+3$

25. Let Mike sell p products. Then Steve sells $2p$ products.

3.2 Evaluating Algebraic Expressions

In this section we will evaluate *algebraic expressions* for given values of the variables contained in the expressions. Here are some simple tips to help you be successful.

Tips for Evaluating Algebraic Expressions.

1. Replace all occurrences of variables in the expression with open parentheses. Leave room between the parentheses to substitute the given value of the variable.
2. Substitute the given values of variables in the open parentheses prepared in the first step.
3. Evaluate the resulting expression according to the *Rules Guiding Order of Operations*.

Let's begin with an example.

You Try It!

EXAMPLE 1. Evaluate the expression $x^2 - 2xy + y^2$ at $x = -3$ and $y = 2$.

If $x = -2$ and $y = -1$, evaluate $x^3 - y^3$.

Solution. Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression $x^2 - 2xy + y^2$ with open parentheses.

$$x^2 - 2xy + y^2 = (\quad)^2 - 2(\quad)(\quad) + (\quad)^2$$

Secondly, replace each variable with its given value, and thirdly, follow the "Rules Guiding Order of Operations" to evaluate the resulting expression.

$x^2 - 2xy + y^2$	Original expression.
$= (-3)^2 - 2(-3)(2) + (2)^2$	Substitute -3 for x and 2 for y.
$= 9 - 2(-3)(2) + 4$	Evaluate exponents first.
$= 9 - (-6)(2) + 4$	Left to right, multiply: $2(-3) = -6$.
$= 9 - (-12) + 4$	Left to right, multiply: $(-6)(2) = -12$.
$= 9 + 12 + 4$	Add the opposite.
$= 25$	Add.

Answer: -7

□

You Try It!

If $a = 3$ and $b = -5$, evaluate $a^2 - b^2$.

EXAMPLE 2. Evaluate the expression $(a-b)^2$ at $a = 3$ and $b = -5$.

Solution. Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression $(a-b)^2$ with open parentheses.

$$(a-b)^2 = ((\quad) - (\quad))^2$$

Secondly, replace each variable with its given value, and thirdly, follow the "Rules Guiding Order of Operations" to evaluate the resulting expression.

$$\begin{aligned}
(a-b)^2 &= ((3) - (-5))^2 && \text{Substitute 3 for } a \text{ and } -5 \text{ for } b. \\
&= (3+5)^2 && \text{Add the opposite: } (3) - (-5) = 3 + 5 \\
&= 8^2 && \text{Simplify inside parentheses: } 3 + 5 = 8 \\
&= 64 && \text{Evaluate exponent: } 8^2 = 64
\end{aligned}$$

Answer: -16

□

You Try It!

If $a = 5$ and $b = -7$, evaluate $2|a| - 3|b|$.

EXAMPLE 3. Evaluate the expression $|a| - |b|$ at $a = 5$ and $b = -7$.

Solution. Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression $|a| - |b|$ with open parentheses.

$$|a| - |b| = |(\quad)| - |(\quad)|$$

Secondly, replace each variable with its given value, and thirdly, follow the "Rules Guiding Order of Operations" to evaluate the resulting expression.

$$\begin{aligned}
|a| - |b| &= |(5)| - |(-7)| && \text{Substitute 5 for } a \text{ and } -7 \text{ for } b. \\
&= 5 - 7 && \text{Absolute values first: } |(5)| = 5 \text{ and } |(-7)| = 7 \\
&= 5 + (-7) && \text{Add the opposite: } 5 - 7 = 5 + (-7). \\
&= -2 && \text{Add: } 5 + (-7) = -2.
\end{aligned}$$

Answer: -11

□

You Try It!

If $a = 5$ and $b = -7$, evaluate $|2a - 3b|$.

EXAMPLE 4. Evaluate the expression $|a - b|$ at $a = 5$ and $b = -7$.

Solution. Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression $|a - b|$ with open parentheses.

$$|a - b| = |(\quad) - (\quad)|$$

Secondly, replace each variable with its given value, and thirdly, follow the "Rules Guiding Order of Operations" to evaluate the resulting expression.

$$\begin{aligned} |a-b| &= |(5)-(-7)| && \text{Substitute 5 for } a \text{ and } -7 \text{ for } b. \\ &= |5+7| && \text{Add the opposite: } 5-(-7)=5+7. \\ &= |12| && \text{Add: } 5+7=12. \\ &= 12 && \text{Take the absolute value: } |12|=12. \end{aligned}$$

Answer: 31

You Try It!

EXAMPLE 5. Evaluate the expression

$$\frac{ad-bc}{a+b}$$

at $a=5$, $b=-3$, $c=2$, and $d=-4$.

Solution. Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression with open parentheses.

$$\frac{ad-bc}{a+b} = \frac{(\quad)(\quad)-(\quad)(\quad)}{(\quad)+(\quad)}$$

Secondly, replace each variable with its given value, and thirdly, follow the "Rules Guiding Order of Operations" to evaluate the resulting expression.

$$\begin{aligned} \frac{ad-bc}{a+b} &= \frac{(5)(-4)-(-3)(2)}{(5)+(-3)} && \text{Substitute: 5 for } a, -3 \text{ for } b, 2 \text{ for } c, -4 \text{ for } d. \\ &= \frac{-20-(-6)}{2} && \text{Numerator: } (5)(-4)=-20, (-3)(2)=-6. \\ & && \text{Denominator: } 5+(-3)=2. \\ &= \frac{-20+6}{2} && \text{Numerator: Add the opposite.} \\ &= \frac{-14}{2} && \text{Numerator: } -20+6=-14. \\ &= -7 && \text{Divide.} \end{aligned}$$

If $a=-7$, $b=-3$, $c=-15$, and $d=-14$, evaluate:

$$\frac{a^2+b^2}{c+d}$$

Answer -2

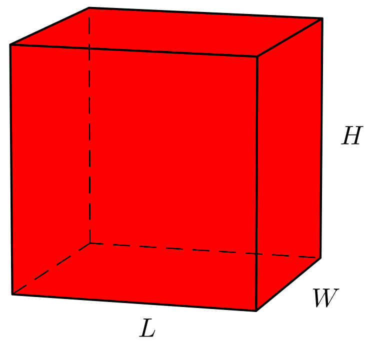
You Try It!

The surface area of the prism pictured in this example is given by the following formula:

$$S = 2(WH + LH + LW)$$

If $L = 12$, $W = 4$, and $H = 6$ feet, respectively, calculate the surface area.

EXAMPLE 6. Pictured below is a rectangular prism.



The volume of the rectangular prism is given by the formula

$$V = LWH,$$

where L is the length, W is the width, and H is the height of the rectangular prism. Find the volume of a rectangular prism having length 12 feet, width 4 feet, and height 6 feet.

Solution. Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of of L, W, and H in the formula

$$V = LWH$$

with open parentheses.

$$V = (\quad)(\quad)(\quad)$$

Next, substitute $12\,\text{ft}$ for L, $4\,\text{ft}$ for W, and $6\,\text{ft}$ for H and simplify.

$$\begin{aligned} V &= (12\,\text{ft})(4\,\text{ft})(6\,\text{ft}) \\ &= 288\,\text{ft}^3 \end{aligned}$$

Hence, the volume of the rectangular prism is 288 cubic feet. □

Answer: 288 square feet.

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-12, evaluate the expression at the given value of x.

1. $-3x^2 - 6x + 3$ at $x = 7$

2. $7x^2 - 7x + 1$ at $x = -8$

3. $-6x - 6$ at $x = 3$

4. $6x - 1$ at $x = -10$

5. $5x^2 + 2x + 4$ at $x = -1$

6. $4x^2 - 9x + 4$ at $x = -3$

7. $-9x - 5$ at $x = -2$

8. $-9x + 12$ at $x = 5$

9. $4x^2 + 2x + 6$ at $x = -6$

10. $-3x^2 + 7x + 4$ at $x = -7$

11. $12x + 10$ at $x = -12$

12. $-6x + 7$ at $x = 11$

In Exercises 13-28, evaluate the expression at the given values of x and y.

13. $|x| - |y|$ at $x = -5$ and $y = 4$

14. $|x| - |y|$ at $x = -1$ and $y = -2$

15. $-5x^2 + 2y^2$ at $x = 4$ and $y = 2$

16. $-5x^2 - 4y^2$ at $x = -2$ and $y = -5$

17. $|x| - |y|$ at $x = 0$ and $y = 2$

18. $|x| - |y|$ at $x = -2$ and $y = 0$

19. $|x - y|$ at $x = 4$ and $y = 5$

20. $|x - y|$ at $x = -1$ and $y = -4$

21. $5x^2 - 4xy + 3y^2$ at $x = 1$ and $y = -4$

22. $3x^2 + 5xy + 3y^2$ at $x = 2$ and $y = -1$

23. $|x - y|$ at $x = 4$ and $y = 4$

24. $|x - y|$ at $x = 3$ and $y = -5$

25. $-5x^2 - 3xy + 5y^2$ at $x = -1$ and $y = -2$

26. $3x^2 - 2xy - 5y^2$ at $x = 2$ and $y = 5$

27. $5x^2 + 4y^2$ at $x = -2$ and $y = -2$

28. $-4x^2 + 2y^2$ at $x = 4$ and $y = -5$

In Exercises 29-40, evaluate the expression at the given value of x.

29. $\dfrac{9 + 9x}{-x}$ at $x = -3$

30. $\dfrac{9 - 2x}{-x}$ at $x = -1$

31. $\dfrac{-8x + 9}{-9 + x}$ at $x = 10$

32. $\dfrac{2x + 4}{1 + x}$ at $x = 0$

33. $\dfrac{-4 + 9x}{7x}$ at $x = 2$

34. $\dfrac{-1 - 9x}{x}$ at $x = -1$

35. $\dfrac{-12 - 7x}{x}$ at $x = -1$

36. $\dfrac{12 + 11x}{3x}$ at $x = -6$

37. $\dfrac{6x - 10}{5 + x}$ at $x = -6$

38. $\dfrac{11x + 11}{-4 + x}$ at $x = 5$

39. $\dfrac{10x+11}{5+x}$ at $x=-4$

40. $\dfrac{6x+12}{-3+x}$ at $x=2$

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41. The formula

$$d = 16t^2$$

gives the distance (in feet) that an object falls from rest in terms of the time t that has elapsed since its release. Find the distance d (in feet) that an object falls in $t = 4$ seconds.

42. The formula

$$d = 16t^2$$

gives the distance (in feet) that an object falls from rest in terms of the time t that has elapsed since its release. Find the distance d (in feet) that an object falls in $t = 24$ seconds.

43. The formula

$$C = \frac{5(F-32)}{9}$$

gives the Celcius temperature C in terms of the Fahrenheit temperature F. Use the formula to find the Celsius temperature ($^\circ$ C) if the Fahrenheit temperature is $F = 230^\circ$ F.

44. The formula

$$C = \frac{5(F-32)}{9}$$

gives the Celcius temperature C in terms of the Fahrenheit temperature F. Use the formula to find the Celsius temperature ($^\circ$ C) if the Fahrenheit temperature is $F = 95^\circ$ F.

45. The Kelvin scale of temperature is used in chemistry and physics. Absolute zero occurs at 0° K, the temperature at which molecules have zero kinetic energy. Water freezes at 273° K and boils at $K = 373^\circ$ K. To change Kelvin temperature to Fahrenheit temperature, we use the formula

$$F = \frac{9(K-273)}{5} + 32.$$

Use the formula to change 28° K to Fahrenheit.

46. The Kelvin scale of temperature is used in chemistry and physics. Absolute zero occurs at 0° K, the temperature at which molecules have zero kinetic energy. Water freezes at 273° K and boils at $K = 373^\circ$ K. To change Kelvin temperature to Fahrenheit temperature, we use the formula

$$F = \frac{9(K-273)}{5} + 32.$$

Use the formula to change 248° K to Fahrenheit.

47. A ball is thrown vertically upward. Its velocity t seconds after its release is given by the formula

$$v = v_0 - gt,$$

where v_0 is its initial velocity, g is the acceleration due to gravity, and v is the velocity of the ball at time t. The acceleration due to gravity is $g = 32$ feet per second per second. If the initial velocity of the ball is $v_0 = 272$ feet per second, find the speed of the ball after $t = 6$ seconds.

48. A ball is thrown vertically upward. Its velocity t seconds after its release is given by the formula

$$v = v_0 - gt,$$

where v_0 is its initial velocity, g is the acceleration due to gravity, and v is the velocity of the ball at time t. The acceleration due to gravity is $g = 32$ feet per second per second. If the initial velocity of the ball is $v_0 = 470$ feet per second, find the speed of the ball after $t = 4$ seconds.

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49. **Even numbers.** Evaluate the expression $2n$ for the following values:

i) $n = 1$

ii) $n = 2$

iii) $n = 3$

iv) $n = -4$

v) $n = -5$

vi) Is the result always an even number? Explain.

50. **Odd numbers.** Evaluate the expression $2n + 1$ for the following values:

i) $n = 1$

ii) $n = 2$

iii) $n = 3$

iv) $n = -4$

v) $n = -5$

vi) Is the result always an odd number? Explain.

❧ ❧ ❧ Answers ☙ ☙ ☙

1. -186

3. -24

5. 7

7. 13

9. 138

11. -134

13. 1

15. -72

17. -2

19. 1

21. 69

23. 0

25. 9

27. 36

29. -6

31. -71

33. 1

35. 5

37. 46

39. -29

41. 256 feet

43. 110 degrees

45. -409° F

47. 80 feet per second

49.
i) 2
ii) 4
iii) 6
iv) -8
v) -10
vi) Yes, the result will always be an even number because 2 will always be a factor of the product $2n$.

3.3 Simplifying Algebraic Expressions

Recall the commutative and associative properties of multiplication.

The Commutative Property of Multiplication. If a and b are any integers, then

$$a \cdot b = b \cdot a, \qquad \text{or equivalently,} \qquad ab = ba.$$

The Associative Property of Multiplication. If a, b, and c are any integers, then

$$(a \cdot b) \cdot c = a \cdot (b \cdot c), \qquad \text{or equivalently,} \qquad (ab)c = a(bc).$$

The commutative property allows us to change the order of multiplication without affecting the product or answer. The associative property allows us to regroup without affecting the product or answer.

You Try It!

EXAMPLE 1. Simplify: $2(3x)$.

Simplify: $-5(7y)$

Solution. Use the associative property to regroup, then simplify.

$$\begin{aligned} 2(3x) &= (2 \cdot 3)x && \text{Regrouping with the associative property.} \\ &= 6x && \text{Simplify: } 2 \cdot 3 = 6. \end{aligned}$$

Answer: $-35y$

□

The statement $2(3x) = 6x$ is an *identity*. That is, the left-hand side and right-hand side of $2(3x) = 6x$ are the same *for all values of* x. Although the derivation in Example 1 should be the proof of this statement, it helps the intuition to check the validity of the statement for one or two values of x.

If $x = 4$, then

$$\begin{aligned} 2(3x) &= 2(3(4)) \\ &= 2(12) \\ &= 24 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 6x &= 6(4) \\ &= 24 \end{aligned}$$

If $x = -5$, then

$$\begin{aligned} 2(3x) &= 2(3(-5)) \\ &= 2(-15) \\ &= -30 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 6x &= 6(-5) \\ &= -30 \end{aligned}$$

The above calculations show that $2(3x) = 6x$ for both $x = 4$ and $x = -5$. Indeed, the statement $2(3x) = 6x$ is true, regardless of what is substituted for x.

You Try It!

Simplify: $(-8a)(5)$

EXAMPLE 2. Simplify: $(-3t)(-5)$.

Solution. In essence, we are multiplying three numbers, -3, t, and -5, but the grouping symbols ask us to multiply the -3 and the t first. The associative and commutative properties allow us to change the order and regroup.

$$\begin{aligned} (-3t)(-5) &= ((-3)(-5))t && \text{Change the order and regroup.} \\ &= 15t && \text{Multiply: } (-3)(-5) = 15. \end{aligned}$$

Answer: $-40a$

□

You Try It!

Simplify: $(-4a)(5b)$

EXAMPLE 3. Simplify: $(-3x)(-2y)$.

Solution. In essence, we are multiplying four numbers, -3, x, -2, and y, but the grouping symbols specify a particular order. The associative and commutative properties allow us to change the order and regroup.

$$\begin{aligned} (-3x)(-2y) &= ((-3)(-2))(xy) && \text{Change the order and regroup.} \\ &= 6xy && \text{Multiply: } (-3)(-2) = 6. \end{aligned}$$

Answer: $-20ab$

□

Speeding Things Up

The meaning of the expression $2 \cdot 3 \cdot 4$ is clear. Parentheses and order of operations are really not needed, as the commutative and associative properties explain that it doesn't matter which of the three numbers you multiply together first.

- You can multiply 2 and 3 first:

$$\begin{aligned} 2 \cdot 3 \cdot 4 &= (2 \cdot 3) \cdot 4 \\ &= 6 \cdot 4 \\ &= 24. \end{aligned}$$

- Or you can multiply 3 and 4 first:

$$\begin{aligned} 2 \cdot 3 \cdot 4 &= 2 \cdot (3 \cdot 4) \\ &= 2 \cdot 12 \\ &= 24. \end{aligned}$$

- Or you can multiply 2 and 4 first:

$$\begin{aligned} 2 \cdot 3 \cdot 4 &= (2 \cdot 4) \cdot 3 \\ &= 8 \cdot 3 \\ &= 24. \end{aligned}$$

So, it doesn't matter which two factors you multiply first.

Of course, this would not be the case if there were a mixture of multiplication and other operators (division, addition, subtraction). Then we would have to strictly follow the "Rules Guiding Order of Operations." But if the only operator is multiplication, the order of multiplication is irrelevant.

Thus, when we see $2(3x)$, as in Example 1, we should think "It's all multiplication and it doesn't matter which two numbers I multiply together first, so I'll multiply the 2 and the 3 and get $2(3x) = 6x$."

Our comments apply equally well to a product of four or more factors. It simply doesn't matter how you group the multiplication. So, in the case of $(-3x)(-2y)$, as in Example 3, find the product of -2 and -3 and multiply the result by the product of x and y. That is, $(-3x)(-2y) = 6xy$.

You Try It!

Simplify: $(-3x)(-2y)(-4z)$

EXAMPLE 4. Simplify: $(2a)(3b)(4c)$.

Solution. The only operator is multiplication, so we can order and group as we please. So, we'll take the product of 2, 3, and 4, and multiply the result by the product of a, b, and c. That is,

$$(2a)(3b)(4c) = 24abc.$$

Answer: $-24xyz$

□

The Distributive Property

Multiplication is distributive with respect to addition.

The Distributive Property. If a, b, and c are any integers, then

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad \text{or equivalently,} \quad a(b + c) = ab + ac.$$

For example, if we follow the "Rules Guiding Order of Operations" and first evaluate the expression inside the parentheses, then

$$\begin{aligned} 3(4 + 5) &= 3(9) && \text{Parentheses first: } 4 + 5 = 9. \\ &= 27. && \text{Multiply: } 3(9) = 27. \end{aligned}$$

But if we “distribute” the 3, we get the same answer.

$$\begin{aligned} 3(4+5) &= 3(4+5) && \text{Each number in parentheses is multiplied by the number 3 outside the parentheses.} \\ &= 3(4)+3(5) \\ &= 12+15 && \text{Multiply first: } 3(4)=12,\ 3(5)=15. \\ &= 27 && \text{Add.} \end{aligned}$$

You Try It!

Use the distributive property to simplify: $2(5z+7)$

EXAMPLE 5. Use the distributive property to simplify: $3(4x+5)$.

Solution. Distribute the 3.

$$\begin{aligned} 3(4x+5) &= 3(4x)+3(5) && \text{Each number in parentheses is multiplied by the number 3 outside the parentheses.} \\ &= 12x+15 && \text{Multiply first: } 3(4x)=12x,\ 3(5)=15. \end{aligned}$$

Answer: $10z+14$

□

Multiplication is also distributive with respect to subtraction.

The Distributive Property. If a, b, and c are any integers, then

$$a\cdot(b-c)=a\cdot b-a\cdot c, \quad \text{or equivalently,} \quad a(b-c)=ab-ac.$$

The application of this form of the distributive property is identical to the first, the only difference being the subtraction symbol.

You Try It!

Use the distributive property to simplify: $7(4a-5)$

EXAMPLE 6. Use the distributive property to simplify: $5(3x-2)$.

Solution. Distribute the 5.

$$\begin{aligned} 5(3x-2) &= 5(3x)-5(2) && \text{Each number in parentheses is multiplied by the number 5 outside the parentheses.} \\ &= 15x-10 && \text{Multiply first: } 5(3x)=15x,\ 5(2)=10. \end{aligned}$$

Answer: $28a-35$

□

You Try It!

Remove parentheses: $-3(4t - 11)$

EXAMPLE 7. Remove parentheses: (a) $-9(2t + 7)$, and (b) $-5(4 - 3y)$.

Solution.

a) Use the distributive property.

$$\begin{aligned} -9(2t+7) &= -9(2t) + (-9)(7) && \text{Distribute multiplication by } -9. \\ &= -18t + (-63) && \text{Multiply: } -9(2t) = -18t \text{ and } -9(7) = -63. \\ &= -18t - 63 && \text{Write the answer in simpler form.} \\ & && \text{Adding } -63 \text{ is the same as} \\ & && \text{subtracting } 63. \end{aligned}$$

b) Use the distributive property.

$$\begin{aligned} -5(4-3y) &= -5(4) - (-5)(3y) && \text{Distribute multiplication by } -5. \\ &= -20 - (-15y) && \text{Multiply: } -5(4) = -20 \\ & && \text{and } (-5)(3y) = -15y. \\ &= -20 + 15y && \text{Write the answer in simpler form.} \\ & && \text{Subtracting } -15y \text{ is the same as} \\ & && \text{adding } 15y. \end{aligned}$$

Answer: $-12t + 33$

□

Writing Mathematics. Example 7 stresses the importance of using as few symbols as possible to write your final answer. Hence, $-18t - 63$ is favored over $-18t + (-63)$ and $-20 + 15y$ is favored over $-20 - (-15y)$. You should always make these final simplifications.

Moving a Bit Quicker

Once you've applied the distributive property to a number of problems, showing all the work as in Example 7, you should try to eliminate some of the steps. For example, consider again Example 7(a). It's not difficult to apply the distributive property without writing down a single step, getting:

$$-9(2t + 7) = -18t - 63.$$

Here's the thinking behind this technique:

1. First, multiply -9 times $2t$, getting $-18t$.

2. Second, multiply -9 times $+7$, getting -63.

Note that this provides exactly the same solution found in Example 7(a).

Let try this same technique on Example 7(b).

$$-5(4-3y)=-20+15y$$

Here's the thinking behind this technique.

1. First, multiply -5 times 4, getting -20.
2. Second, multiply -5 times $-3y$, getting $+15y$.

Note that this provides exactly the same solution found in Example 7(b).

Extending the Distributive Property

Suppose that we add an extra term inside the parentheses.

Distributive Property. If a, b, c, and d are any integers, then

$$a(b+c+d)=ab+ac+ad.$$

Note that we "distributed" the a times each term inside the parentheses. Indeed, if we added still another term inside the parentheses, we would "distribute" a times that term as well.

You Try It!

Remove parentheses: $-3(4a-5b+7)$

EXAMPLE 8. Remove parentheses: $-5(2x-3y+8)$.

Solution. We will use the "quicker" technique, "distributing" -5 times each term in the parentheses mentally.

$$-5(2x-3y+8)=-10x+15y-40$$

Here is our thought process:

1. First, multiply -5 times $2x$, getting $-10x$.
2. Second, multiply -5 times $-3y$, getting $+15y$.
3. Third, multiply -5 times $+8$, getting -40.

Answer: $-12a+15b-21$

□

You Try It!

Remove parentheses: $-2(-2x + 4y - 5z - 11)$

EXAMPLE 9. Remove parentheses: $-4(-3a + 4b - 5c + 12)$.

Solution. We will use the "quicker" technique, "distributing" -4 times each term in the parentheses mentally.

$$-4(-3a + 4b - 5c + 12) = 12a - 16b + 20c - 48$$

Here is our thought process:

1. First, multiply -4 times $-3a$, getting $12a$.
2. Second, multiply -4 times $+4b$, getting $-16b$.
3. Third, multiply -4 times $-5c$, getting $+20c$.
4. Fourth, multiply -4 times $+12$, getting -48.

Answer: $4x - 8y + 10z + 22$

□

Distributing a Negative

It is helpful to recall that negating is equivalent to multiplying by -1.

Multiplying by -1. Let a be any integer, then

$$(-1)a = -a \quad \text{and} \quad -a = (-1)a.$$

We can use this fact, combined with the distributive property, to negate a sum.

You Try It!

Remove parentheses: $-(x + 2y)$

EXAMPLE 10. Remove parentheses: $-(a + b)$.

Solution. Change the negative symbol into multiplying by -1, then distribute the -1.

$$\begin{aligned} -(a+b) &= (-1)(a+b) && \text{Negating is equivalent to multiplying by } -1. \\ &= -a - b && \text{Distribute the } -1. \end{aligned}$$

We chose to use the "quicker" technique of "distributing" the -1. Here is our thinking:

1. Multiply -1 times a, getting $-a$.
2. Multiply -1 times $+b$, getting $-b$.

Answer: $-x - 2y$

You Try It!

Remove parentheses: $-(4a - 3c)$

EXAMPLE 11. Remove parentheses: $-(a - b)$.

Solution. Change the negative symbol into multiplying by -1, then distribute the -1.

$$\begin{aligned} -(a-b) &= (-1)(a-b) && \text{Negating is equivalent to multiplying by } -1. \\ &= -a + b && \text{Distribute the } -1. \end{aligned}$$

We chose to use the "quicker" technique of "distributing" the -1. Here is our thinking:

1. Multiply -1 times a, getting $-a$.
2. Multiply -1 times $-b$, getting $+b$.

Answer: $-4a + 3c$

The results in Example 10 and Example 11 show us how to negate a sum: Simply negate each term of the sum. Positive terms change to negative, negative terms turn to positive.

Negating a Sum. To negate a sum, simply negate each term of the sum. For example, if a and b are integers, then

$$-(a+b) = -a - b \qquad \text{and} \qquad -(a-b) = -a + b.$$

You Try It!

Remove parentheses: $-(5 - 2x + 4y - 5z)$

EXAMPLE 12. Remove parentheses: $-(5 - 7u + 3t)$.

Solution. Simply negate each term in the parentheses.

$$-(5 - 7u + 3t) = -5 + 7u - 3t.$$

Answer: $-5 + 2x - 4y + 5z$

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-20, use the associative and commutative properties of multiplication to simplify the expression.

1. $10(-4x)$

2. $7(-8x)$

3. $(-10x)(-3)$

4. $(-5x)(-8)$

5. $-5(3x)$

6. $9(6x)$

7. $(-4x)10$

8. $(-10x)(-6)$

9. $(5x)3$

10. $(3x)3$

11. $(5x)10$

12. $(-2x)(-10)$

13. $-9(-7x)$

14. $-10(5x)$

15. $6(2x)$

16. $3(-10x)$

17. $-8(-9x)$

18. $3(-3x)$

19. $(6x)7$

20. $(-8x)(-5)$

In Exercises 21-44, simplify the expression.

21. $8(7x + 8)$

22. $-2(5x + 5)$

23. $9(-2 + 10x)$

24. $-9(4 + 9x)$

25. $-(-2x + 10y - 6)$

26. $-(-6y + 9x - 7)$

27. $2(10 + x)$

28. $2(10 - 6x)$

29. $3(3 + 4x)$

30. $3(4 + 6x)$

31. $-(-5 - 7x + 2y)$

32. $-(4x - 8 - 7y)$

33. $4(-6x + 7)$

34. $6(4x + 9)$

35. $4(8x - 9)$

36. $10(-10x + 1)$

37. $-(4 - 2x - 10y)$

38. $-(-4x + 6 - 8y)$

39. $-(-5x + 1 + 9y)$

40. $-(-10 - 5x - 4y)$

41. $-(6x + 2 - 10y)$

42. $-(6x + 4 - 10y)$

43. $-(-3y - 4 + 4x)$

44. $-(-7 - 10x + 7y)$

❧ ❧ ❧ Answers ☙ ☙ ☙

1. $-40x$

3. $30x$

5. $-15x$

7. $-40x$

9. $15x$

11. $50x$

13. $63x$

15. $12x$

17. $72x$

19. $42x$

21. $56x+64$

23. $-18+90x$

25. $2x-10y+6$

27. $20+2x$

29. $9+12x$

31. $5+7x-2y$

33. $-24x+28$

35. $32x-36$

37. $-4+2x+10y$

39. $5x-1-9y$

41. $-6x-2+10y$

43. $3y+4-4x$

3.4 Combining Like Terms

We begin our discussion with the definition of a *term*.

Term. A *term* is a single number or variable, or it can be the product of a number (called its *coefficient*) and one or more variables (called its *variable part*). The terms in an algebraic expression are separated by *addition* symbols.

You Try It!

EXAMPLE 1. Identify the terms in the algebraic expression

$$3x^2 + 5xy + 9y^2 + 12.$$

For each term, identify its coefficient and variable part.

How many terms are in the algebraic expression $3x^2 + 2xy - 3y^2$?

Solution. In tabular form, we list each term of the expression $3x^2 + 5xy + 9y^2 + 12$, its coefficient, and its variable part.

Term	**Coefficient**	**Variable Part**
$3x^2$	3	x^2
$5xy$	5	xy
$9y^2$	9	y^2
12	12	None

Answer: 3

□

You Try It!

EXAMPLE 2. Identify the terms in the algebraic expression

$$a^3 - 3a^2b + 3ab^2 - b^3.$$

For each term, identify its coefficient and variable part.

How many terms are in the algebraic expression $11 - a^2 - 2ab + 3b^2$?

Solution. The first step is to write each difference as a sum, because the terms of an expression are defined above to be those items separated by addition symbols.

$$a^3 + (-3a^2b) + 3ab^2 + (-b^3)$$

In tabular form, we list each term of the expression $a^3+(-3a^2b)+3ab^2+(-b^3)$, its coefficient, and its variable part.

Term	**Coefficient**	**Variable Part**
a^3	1	a^3
$-3a^2b$	-3	a^2b
$3ab^2$	3	ab^2
$-b^3$	-1	b^3

Answer: 4

□

Like Terms

We define what is meant by "like terms" and "unlike terms."

Like and Unlike Terms. The variable parts of two terms determine whether the terms are *like terms* or *unlike terms*.

Like Terms. Two terms are called *like terms* if they have identical variable parts, which means that the terms must contain the same variables raised to the same exponential powers.

Unlike Terms. Two terms are called *unlike terms* if their variable parts are different.

You Try It!

Are $-3xy$ and $11xy$ *like* or *unlike* terms?

EXAMPLE 3. Classify each of the following pairs as either *like terms* or *unlike terms*: (a) $3x$ and $-7x$, (b) $2y$ and $3y^2$, (c) $-3t$ and $5u$, and (d) $-4a^3$ and $3a^3$.

Solution. Like terms must have *identical* variable parts.

a) $3x$ and $-7x$ have identical variable parts. They are "like terms."

b) $2y$ and $3y^2$ do **not** have identical variable parts (the exponents differ). They are "unlike terms."

c) $-3t$ and $5u$ do **not** have identical variable parts (different variables). They are "unlike terms."

d) $-4a^3$ and $3a^3$ have identical variable parts. They are "like terms."

Answer: *Like* terms

□

Combining Like Terms

When using the distributive property, it makes no difference whether the multiplication is on the left or the right, one still distributes the multiplication times each term in the parentheses.

Distributive Property. If a, b, and c are integers, then

$$a(b+c) = ab + ac \qquad \text{and} \qquad (b+c)a = ba + ca.$$

In either case, you distribute a times each term of the sum.

"Like terms" can be combined and simplified. The tool used for combining like terms is the distributive property. For example, consider the expression $3y + 7y$, composed of two "like terms" with a common variable part. We can use the distributive property and write

$$3y + 7y = (3+7)y.$$

Note that we are using the distributive property in reverse, "factoring out" the common variable part of each term. *Checking our work*, note that if we redistribute the variable part y times each term in the parentheses, we are returned to the original expression $3y + 7y$.

You Try It!

Simplify: $-8z - 11z$

EXAMPLE 4. Use the distributive property to combine like terms (if possible) in each of the following expressions: (a) $-5x^2 - 9x^2$, (b) $-5ab + 7ab$, (c) $4y^3 - 7y^2$, and (d) $3xy^2 - 7xy^2$.

Solution. If the terms are "like terms," you can use the distributive property to "factor out" the common variable part.

a) Factor out the common variable part x^2.

$$\begin{aligned} -5x^2 - 9x^2 &= (-5-9)x^2 && \text{Use the distributive property.} \\ &= -14x^2 && \text{Simplify: } -5-9 = -5+(-9) = -14. \end{aligned}$$

b) Factor out the common variable part ab.

$$\begin{aligned} -5ab + 7ab &= (-5+7)ab && \text{Use the distributive property.} \\ &= 2ab && \text{Simplify: } -5+7 = 2. \end{aligned}$$

c) The terms in the expression $4y^3 - 7y^2$ have different variable parts (the exponents are different). These are "unlike terms" and cannot be combined.

d) Factor out the common variable part xy^2.

$$\begin{aligned} 3xy^2 - 7xy^2 &= (3-7)xy^2 && \text{Use the distributive property.} \\ &= -4xy^2 && \text{Simplify: } 3-7 = 3+(-7) = -4. \end{aligned}$$

Answer: $-19z$

Speeding Things Up a Bit

Once you've written out all the steps for combining like terms, like those shown in Example 4, you can speed things up a bit by following this rule:

Combining Like Terms. To combine like terms, simply add their coefficients and keep the common variable part.

Thus for example, when presented with the sum of two like terms, such as in $5x + 8x$, simply add the coefficients and repeat the common variable part; that is, $5x + 8x = 13x$.

You Try It!

Combine: $-3x^2 - 4x^2$

EXAMPLE 5. Combine like terms: (a) $-9y - 8y$, (b) $-3y^5 + 4y^5$, and (c) $-3u^2 + 2u^2$.

Solution.

a) Add the coefficients and repeat the common variable part. Therefore,

$$-9y - 8y = -17y.$$

b) Add the coefficients and repeat the common variable part. Therefore,

$$-3y^5 + 4y^5 = 1y^5.$$

However, note that $1y^5 = y^5$. Following the rule that the final answer should use as few symbols as possible, a better answer is $-3y^5 + 4y^5 = y^5$.

c) Add the coefficients and repeat the common variable part. Therefore,

$$-3u^2 + 2u^2 = (-1)u^2.$$

However, note that $(-1)u^2 = -u^2$. Following the rule that the final answer should use as few symbols as possible, a better answer is $-3u^2+2u^2 = -u^2$.

Answer: $-7x^2$

Simplify

A frequently occurring instruction asks the reader to *simplify* an expression.

Simplify. The instruction *simplify* is a generic term that means "try to write the expression in its most compact form, using the fewest symbols possible."

One way you can accomplish this goal is by combining like terms when they are present.

You Try It!

EXAMPLE 6. Simplify: $2x + 3y - 5x + 8y$.

Simplify: $-3a + 4b - 7a - 9b$

Solution. Use the commutative property to reorder terms and the associative and distributive properties to regroup and combine like terms.

$$\begin{aligned} 2x + 3y - 5x + 8y &= (2x - 5x) + (3y + 8y) && \text{Reorder and regroup.} \\ &= -3x + 11y && \text{Combine like terms:} \\ & && 2x - 5x = -3x \text{ and } 3y + 8y = 11y. \end{aligned}$$

Alternate solution. Of course, you do not need to show the regrouping step. If you are more comfortable combining like terms in your head, you are free to present your work as follows:

$$2x + 3y - 5x + 8y = -3x + 11y.$$

Answer: $-10a - 5b$

□

You Try It!

EXAMPLE 7. Simplify: $-2x - 3 - (3x + 4)$.

Simplify: $-9a - 4 - (4a - 8)$

Solution. First, distribute the negative sign.

$$-2x - 3 - (3x + 4) = -2x - 3 - 3x - 4 \qquad -(3x + 4) = -3x - 4.$$

Next, use the commutative property to reorder, then the associative property to regroup. Then combine like terms.

$$\begin{aligned} &= (-2x - 3x) + (-3 - 4) && \text{Reorder and regroup.} \\ &= -5x + (-7) && \text{Combine like terms:} \\ & && -2x - 3x = -5x. \\ &= -5x - 7 && \text{Simplify:} \\ & && -5x + (-7) = -5x - 7. \end{aligned}$$

Alternate solution. You may skip the second step if you wish, simply combining like terms mentally. That is, it is entirely possible to order your work as follows:

$$\begin{aligned} -2x - 3 - (3x + 4) &= -2x - 3 - 3x - 4 && \text{Distribute negative sign.} \\ &= -5x - 7 && \text{Combine like terms.} \end{aligned}$$

Answer: $-13a + 4$

□

You Try It!

Simplify:

$-2(3a - 4) - 2(5 - a)$

EXAMPLE 8. Simplify: $2(5 - 3x) - 4(x + 3)$.

Solution. Use the distributive property to expand, then use the commutative and associative properties to group the like terms and combine them.

$$\begin{aligned} 2(5 - 3x) - 4(x + 3) &= 10 - 6x - 4x - 12 && \text{Use the distributive property.} \\ &= (-6x - 4x) + (10 - 12) && \text{Group like terms.} \\ &= -10x - 2 && \text{Combine like terms: } -6x - 4x = -10x \text{ and } 10 - 12 = -2. \end{aligned}$$

Alternate solution. You may skip the second step if you wish, simply combining like terms mentally. That is, it is entirely possible to order your work as follows:

$$\begin{aligned} 2(5 - 3x) - 4(x + 3) &= 10 - 6x - 4x - 12 && \text{Distribute.} \\ &= -10x - 2 && \text{Combine like terms.} \end{aligned}$$

Answer: $-4a - 2$

□

You Try It!

Simplify:
$(a^2 - 2ab) - 2(3ab + a^2)$

EXAMPLE 9. Simplify: $-8(3x^2y - 9xy) - 8(-7x^2y - 8xy)$.

Solution. We will proceed a bit quicker with this solution, using the distributive property to expand, then combining like terms mentally.

$$\begin{aligned} -8(3x^2y - 9xy) - 8(-7x^2y - 8xy) &= -24x^2y + 72xy + 56x^2y + 64xy \\ &= 32x^2y + 136xy \end{aligned}$$

Answer: $-a^2 - 8ab$

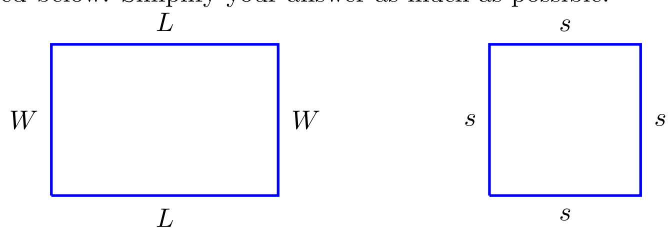
□

Applications

We can simplify a number of useful formulas by combining like terms.

You Try It!

EXAMPLE 10. Find the perimeter P of the (a) rectangle and (b) square pictured below. Simplify your answer as much as possible.



Solution. The perimeter of any polygonal figure is the sum of the lengths of its sides.

a) To find the perimeter P of the rectangle, sum its four sides.

$$P = L + W + L + W.$$

Combine like terms.

$$P = 2L + 2W.$$

b) To find the perimeter P of the square, sum its four sides.

$$P = s + s + s + s.$$

Combine like terms.

$$P = 4s.$$

□

A regular hexagon has six equal sides, each with length x. Find its perimeter in terms of x.

Answer: $P = 6x$

Sometimes it is useful to replace a variable with an expression containing another variable.

You Try It!

EXAMPLE 11. The length of a rectangle is three feet longer than twice its width. Find the perimeter P of the rectangle in terms of its width alone.

Solution. From the previous problem, the perimeter of the rectangle is given by

$$P = 2L + 2W, \tag{3.1}$$

where L and W are the length and width of the rectangle, respectively. This equation gives the perimeter in terms of its length and width, but we're asked to get the perimeter in terms of the width alone.

However, we're also given the fact that the length is three feet longer than twice the width.

The length L of a rectangle is 5 meters longer than twice its width W. Find the perimeter P of the rectangle in terms of its width W.

Length	is	Three Feet	longer than	Twice the Width
L	$=$	3	$+$	$2W$

Because $L = 3+2W$, we can replace L with $3+2W$ in the perimeter equation 3.1.

$$P = 2L + 2W$$
$$P = 2(3 + 2W) + 2W$$

Use the distributive property, then combine like terms.

$$P = 6 + 4W + 2W$$
$$P = 6 + 6W.$$

Answer: $P = 6W + 10$

This last equation gives the perimeter P in terms of the width W alone. □

You Try It!

The width W of a rectangle is 5 feet less than twice its width L. Find the perimeter P of the rectangle in terms of its length L.

EXAMPLE 12. The width of a rectangle is two feet less than its length. Find the perimeter P of the rectangle in terms of its length alone.

Solution. Again, the perimeter of a rectangle is given by the equation

$$P = 2L + 2W, \tag{3.2}$$

where L and W are the length and width of the rectangle, respectively. This equation gives the perimeter in terms of its length and width, but we're asked to get the perimeter in terms of the length alone.

However, we're also given the fact that the width is two feet less than the length.

Width	is	Length	minus	Two feet
W	$=$	L	$-$	2

Because $W = L-2$, we can replace W with $L-2$ in the perimeter equation 3.2.

$$P = 2L + 2W$$
$$P = 2L + 2(L - 2)$$

Use the distributive property, then combine like terms.

$$P = 2L + 2L - 4$$
$$P = 4L - 4.$$

Answer: $P = 6L - 10$

This last equation gives the perimeter P in terms of the length L alone. □

Exercises

In Exercises 1-16, combine like terms by first using the distributive property to factor out the common variable part, and then simplifying.

1. $17xy^2 + 18xy^2 + 20xy^2$

2. $13xy - 3xy + xy$

3. $-8xy^2 - 3xy^2 - 10xy^2$

4. $-12xy - 2xy + 10xy$

5. $4xy - 20xy$

6. $-7y^3 + 15y^3$

7. $12r - 12r$

8. $16s - 5s$

9. $-11x - 13x + 8x$

10. $-9r - 10r + 3r$

11. $-5q + 7q$

12. $17n + 15n$

13. $r - 13r - 7r$

14. $19m + m + 15m$

15. $3x^3 - 18x^3$

16. $13x^2y + 2x^2y$

In Exercises 17-32, combine like terms by first rearranging the terms, then using the distributive property to factor out the common variable part, and then simplifying.

17. $-8 + 17n + 10 + 8n$

18. $11 + 16s - 14 - 6s$

19. $-2x^3 - 19x^2y - 15x^2y + 11x^3$

20. $-9x^2y - 10y^3 - 10y^3 + 17x^2y$

21. $-14xy - 2x^3 - 2x^3 - 4xy$

22. $-4x^3 + 12xy + 4xy - 12x^3$

23. $-13 + 16m + m + 16$

24. $9 - 11x - 8x + 15$

25. $-14x^2y - 2xy^2 + 8x^2y + 18xy^2$

26. $-19y^2 + 18y^3 - 5y^2 - 17y^3$

27. $-14x^3 + 16xy + 5x^3 + 8xy$

28. $-16xy + 16y^2 + 7xy + 17y^2$

29. $9n + 10 + 7 + 15n$

30. $-12r + 5 + 17 + 17r$

31. $3y + 1 + 6y + 3$

32. $19p + 6 + 8p + 13$

In Exercises 33-56, simplify the expression by first using the distributive property to expand the expression, and then rearranging and combining like terms mentally.

33. $-4(9x^2y + 8) + 6(10x^2y - 6)$

34. $-4(-4xy + 5y^3) + 6(-5xy - 9y^3)$

35. $3(-4x^2 + 10y^2) + 10(4y^2 - x^2)$

36. $-7(-7x^3 + 6x^2) - 7(-10x^2 - 7x^3)$

37. $-s + 7 - (-1 - 3s)$

38. $10y - 6 - (-10 - 10y)$

39. $-10q - 10 - (-3q + 5)$

40. $-2n + 10 - (7n - 1)$

41. $7(8y + 7) - 6(8 - 7y)$

42. $-6(-5n - 4) - 9(3 + 4n)$

43. $7(10x^2 - 8xy^2) - 7(9xy^2 + 9x^2)$

44. $10(8x^2y - 10xy^2) + 3(8xy^2 + 2x^2y)$

45. $-2(6 + 4n) + 4(-n - 7)$

46. $-6(-2 - 6m) + 5(-9m + 7)$

47. $8 - (4 + 8y)$

48. $-1 - (8 + s)$

49. $-8(-n + 4) - 10(-4n + 3)$

50. $3(8r - 7) - 3(2r - 2)$

51. $-5 - (10p + 5)$

52. $-1 - (2p - 8)$

53. $7(1 + 7r) + 2(4 - 5r)$

54. $(5 - s) + 10(9 + 5s)$

55. $-2(-5 - 8x^2) - 6(6)$

56. $8(10y^2 + 3x^3) - 5(-7y^2 - 7x^3)$

---

57. The length L of a rectangle is 2 feet longer than 6 times its width W. Find the perimeter of the rectangle in terms of its width alone.

58. The length L of a rectangle is 7 feet longer than 6 times its width W. Find the perimeter of the rectangle in terms of its width alone.

59. The width W of a rectangle is 8 feet shorter than its length L. Find the perimeter of the rectangle in terms of its length alone.

60. The width W of a rectangle is 9 feet shorter than its length L. Find the perimeter of the rectangle in terms of its length alone.

61. The length L of a rectangle is 9 feet shorter than 4 times its width W. Find the perimeter of the rectangle in terms of its width alone.

62. The length L of a rectangle is 2 feet shorter than 6 times its width W. Find the perimeter of the rectangle in terms of its width alone.

❧ ❧ ❧ Answers ☙ ☙ ☙

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1. $55xy^2$

3. $-21xy^2$

5. $-16xy$

7. 0

9. $-16x$

11. $2q$

13. $-19r$

15. $-15x^3$

17. $2 + 25n$

19. $9x^3 - 34x^2y$

21. $-18xy - 4x^3$

23. $3 + 17m$

25. $-6x^2y + 16xy^2$

27. $-9x^3 + 24xy$

29. $24n + 17$

31. $9y + 4$

33. $24x^2y - 68$

35. $-22x^2 + 70y^2$

37. $2s + 8$

39. $-7q - 15$

41. $98y + 1$

43. $7x^2 - 119xy^2$

45. $-40 - 12n$

47. $4 - 8y$

49. $48n - 62$

51. $-10 - 10p$

53. $15 + 39r$

55. $-26 + 16x^2$

57. $4 + 14W$

59. $4L - 16$

61. $10W - 18$

3.5 Solving Equations Involving Integers II

We return to solving equations involving integers, only this time the equations will be a bit more advanced, requiring the use of the distributive property and skill at combining like terms. Let's begin.

You Try It!

Solve for x:

$$-6x - 5x = 22$$

EXAMPLE 1. Solve for x: $7x - 11x = 12$.

Solution. Combine like terms.

$$\begin{aligned} 7x - 11x &= 12 && \text{Original equation.} \\ -4x &= 12 && \text{Combine like terms: } 7x - 11x = -4x. \end{aligned}$$

To undo the effect of multiplying by -4, divide both sides of the last equation by -4.

$$\begin{aligned} \frac{-4x}{-4} &= \frac{12}{-4} && \text{Divide both sides by } -4. \\ x &= -3 && \text{Simplify: } 12/(-4) = -3. \end{aligned}$$

Check. Substitute -3 for x in the original equation.

$$\begin{aligned} 7x - 11x &= 12 && \text{Original equation.} \\ 7(-3) - 11(-3) &= 12 && \text{Substitute } -3 \text{ for } x. \\ -21 + 33 &= 12 && \text{On the left, multiply first.} \\ 12 &= 12 && \text{On the left, add.} \end{aligned}$$

Because the last line of the check is a true statement, -3 is a solution of the original equation.

Answer: $x = -2$

□

You Try It!

Solve for x:

$$11 = 3x - (1 - x)$$

EXAMPLE 2. Solve for x: $12 = 5x - (4 + x)$.

Solution. To take the negative of a sum, negate each term in the sum (change each term to its opposite). Thus, $-(4 + x) = -4 - x$.

$$\begin{aligned} 12 &= 5x - (4 + x) && \text{Original equation.} \\ 12 &= 5x - 4 - x && -(4 + x) = -4 - x. \\ 12 &= 4x - 4 && \text{Combine like terms: } 5x - x = 4x. \end{aligned}$$

To undo the effect of subtracting 4, add 4 to both sides of the last equation.

$$\begin{aligned} 12 + 4 &= 4x - 4 + 4 && \text{Add 4 to both sides.} \\ 16 &= 4x && \text{Simplify both sides.} \end{aligned}$$

To undo the effect of multiplying by 4, divide both sides of the last equation by 4.

$$\frac{16}{4} = \frac{4x}{4}$$ Divide both sides by 4.

$$4 = x$$ Simplify: $16/4 = 4$.

Check. Substitute 4 for x in the original equation.

$12 = 5x - (4 + x)$ Original equation.

$12 = 5(4) - (4 + 4)$ Substitute 4 for x.

$12 = 20 - 8$ On the right, $5(4) = 20$ and evaluate parentheses: $4 + 4 = 8$.

$12 = 12$ Simplify.

Because the last line of the check is a true statement, 4 is a solution of the original equation.

Answer: $x = 3$

Variables on Both Sides

Variables can occur on both sides of the equation.

Goal. Isolate the terms containing the variable you are solving for on one side of the equation.

You Try It!

Solve for x:

$$4x - 3 = x$$

EXAMPLE 3. Solve for x: $5x = 3x - 18$.

Solution. To isolate the variables on one side of the equation, subtract $3x$ from both sides of the equation and simplify.

$5x = 3x - 18$ Original equation.

$5x - 3x = 3x - 18 - 3x$ Subtract $3x$ from both sides.

$2x = -18$ Combine like terms: $5x - 3x = 2x$ and $3x - 3x = 0$.

Note that the variable is now isolated on the left-hand side of the equation. To undo the effect of multiplying by 2, divide both sides of the last equation by 2.

$$\frac{2x}{2} = \frac{-18}{2}$$ Divide both sides by 2.

$$x = -9$$ Simplify: $-18/2 = -9$.

Check. Substitute -9 for x in the original equation.

$$\begin{aligned} 5x &= 3x - 18 && \text{Original equation.} \\ 5(-9) &= 3(-9) - 18 && \text{Substitute } -9 \text{ for } x. \\ -45 &= -27 - 18 && \text{Multiply first on both sides.} \\ -45 &= -45 && \text{Subtract on the right: } -27 - 18 = -45. \end{aligned}$$

Because the last line of the check is a true statement, -9 is a solution of the original equation.

Answer: $x = 1$

□

You Try It!

Solve for x:

$$7x = 18 + 9x$$

EXAMPLE 4. Solve for x: $5x = 3 + 6x$.

Solution. To isolate the variables on one side of the equation, subtract $6x$ from both sides of the equation and simplify.

$$\begin{aligned} 5x &= 3 + 6x && \text{Original equation.} \\ 5x - 6x &= 3 + 6x - 6x && \text{Subtract } 6x \text{ from both sides.} \\ -x &= 3 && \text{Combine like terms: } 5x - 6x = -x \\ & && \text{and } 6x - 6x = 0. \end{aligned}$$

Note that the variable is now isolated on the left-hand side of the equation.

There are a couple of ways we can finish this solution. Remember, $-x$ is the same as $(-1)x$, so we could undo the effects of multiplying by -1 by dividing both sides of the equation by -1. Multiplying both sides of the equation by -1 will work equally well. But perhaps the easiest way to proceed is to simply negate both sides of the equation.

$$\begin{aligned} -(-x) &= -3 && \text{Negate both sides.} \\ x &= -3 && \text{Simplify: } -(-x) = x. \end{aligned}$$

Check. Substitute -3 for x in the original equation.

$$\begin{aligned} 5x &= 3 + 6x && \text{Original equation.} \\ 5(-3) &= 3 + 6(-3) && \text{Substitute } -3 \text{ for } x. \\ -15 &= 3 - 18 && \text{Multiply first on both sides.} \\ -15 &= -15 && \text{Subtract on the right: } 3 - 18 = -15. \end{aligned}$$

Because the last line of the check is a true statement, -3 is a solution of the original equation.

Answer: $x = -9$

□

Dealing with −x. If your equation has the form

$$-x = c,$$

where c is some integer, note that this is equivalent to the equation $(-1)x = c$. Therefore, dividing both sides by −1 will produce a solution for x. Multiplying both sides by −1 works equally well. However, perhaps the easiest thing to do is negate each side, producing

$$-(-x) = -c, \quad \text{which is equivalent to} \quad x = -c.$$

You Try It!

EXAMPLE 5. Solve for x: $6x - 5 = 12x + 19$.

Solve for x:

$$2x + 3 = 18 - 3x$$

Solution. To isolate the variables on one side of the equation, subtract $12x$ from both sides of the equation and simplify.

$6x - 5 = 12x + 19$	Original equation.
$6x - 5 - 12x = 12x + 19 - 12x$	Subtract $12x$ from both sides.
$-6x - 5 = 19$	Combine like terms: $6x - 12x = -6x$ and $12x - 12x = 0$.

Note that the variable is now isolated on the left-hand side of the equation. Next, to "undo" subtracting 5, add 5 to both sides of the equation.

$-6x - 5 + 5 = 19 + 5$	Add 5 to both sides.
$-6x = 24$	Simplify: $-5 + 5 = 0$ and $19 + 5 = 24$.

Finally, to "undo" multiplying by −6, divide both sides of the equation by −6.

$\dfrac{-6x}{-6} = \dfrac{24}{-6}$	Divide both sides by −6.
$x = -4$	Simplify: $24/(-6) = -4$.

Check. Substitute −4 for x in the original equation.

$6x - 5 = 12x + 19$	Original equation.
$6(-4) - 5 = 12(-4) + 19$	Substitute −4 for x.
$-24 - 5 = -48 + 19$	Multiply first on both sides.
$-29 = -29$	Add: $-24 - 5 = -29$ and $-48 + 19 = -29$.

Because the last line of the check is a true statement, −4 is a solution of the original equation.

Answer: $x = 3$

□

You Try It!

Solve for x:

$$3(2x-4)-2(5-x)=18$$

EXAMPLE 6. Solve for x: $2(3x+2)-3(4-x)=x+8$.

Solution. Use the distributive property to remove parentheses on the left-hand side of the equation.

$$\begin{aligned}
2(3x+2)-3(4-x) &= x+8 && \text{Original equation.}\\
6x+4-12+3x &= x+8 && \text{Use the distributive property.}\\
9x-8 &= x+8 && \text{Combine like terms: } 6x+3x=9x\\
& && \text{and } 4-12=-8.
\end{aligned}$$

Isolate the variables on the left by subtracting x from both sides of the equation.

$$\begin{aligned}
9x-8-x &= x+8-x && \text{Subtract } x \text{ from both sides.}\\
8x-8 &= 8 && \text{Combine like terms: } 9x-x=8x\\
& && \text{and } x-x=0.
\end{aligned}$$

Note that the variable is now isolated on the left-hand side of the equation. Next, to "undo" subtracting 8, add 8 to both sides of the equation.

$$\begin{aligned}
8x-8+8 &= 8+8 && \text{Add 8 to both sides.}\\
8x &= 16 && \text{Simplify: } -8+8=0 \text{ and } 8+8=16.
\end{aligned}$$

Finally, to "undo" multiplying by 8, divide both sides of the equation by 8.

$$\begin{aligned}
\frac{8x}{8} &= \frac{16}{8} && \text{Divide both sides by 8.}\\
x &= 2 && \text{Simplify: } 16/8=2.
\end{aligned}$$

Check. Substitute 2 for x in the original equation.

$$\begin{aligned}
2(3x+2)-3(4-x) &= x+8 && \text{Original equation.}\\
2(3(2)+2)-3(4-2) &= 2+8 && \text{Substitute 2 for } x.\\
2(6+2)-3(2) &= 10 && \text{Work parentheses on left, add on the right.}\\
2(8)-3(2) &= 10 && \text{Add in parentheses on left.}\\
16-6 &= 10 && \text{Multiply first on left.}\\
10 &= 10 && \text{Subtract on left.}
\end{aligned}$$

Because the last line of the check is a true statement, 2 is a solution of the original equation.

Answer: $x=5$

□

❧ ❧ ❧ Exercises ☙ ☙ ☙

In Exercises 1-16, solve the equation.

1. $-9x + x = -8$

2. $4x - 5x = -3$

3. $-4 = 3x - 4x$

4. $-6 = -5x + 7x$

5. $27x + 51 = -84$

6. $-20x + 46 = 26$

7. $9 = 5x + 9 - 6x$

8. $-6 = x + 3 - 4x$

9. $0 = -18x + 18$

10. $0 = -x + 71$

11. $41 = 28x + 97$

12. $-65 = -x - 35$

13. $8x - 8 - 9x = -3$

14. $6x + 7 - 9x = 4$

15. $-85x + 85 = 0$

16. $17x - 17 = 0$

In Exercises 17-34, solve the equation.

17. $-6x = -5x - 9$

18. $-5x = -3x - 2$

19. $6x - 7 = 5x$

20. $3x + 8 = -5x$

21. $4x - 3 = 5x - 1$

22. $x - 2 = 9x - 2$

23. $-3x + 5 = 3x - 1$

24. $-5x + 9 = -4x - 3$

25. $-5x = -3x + 6$

26. $3x = 4x - 6$

27. $2x - 2 = 4x$

28. $6x - 4 = 2x$

29. $-6x + 8 = -2x$

30. $4x - 9 = 3x$

31. $6x = 4x - 4$

32. $-8x = -6x + 8$

33. $-8x + 2 = -6x + 6$

34. $-3x + 6 = -2x - 5$

In Exercises 35-52, solve the equation.

35. $1 - (x - 2) = -3$

36. $1 - 8(x - 8) = 17$

37. $-7x + 6(x + 8) = -2$

38. $-8x + 4(x + 7) = -12$

39. $8(-6x - 1) = -8$

40. $-7(-2x - 4) = -14$

41. $-7(-4x - 6) = -14$

42. $-2(2x + 8) = -8$

43. $2 - 9(x - 5) = -16$

44. $7 - 2(x + 4) = -1$

45. $7x + 2(x + 9) = -9$

46. $-8x + 7(x - 2) = -14$

47. $2(-x + 8) = 10$

48. $2(-x - 2) = 10$

49. $8 + 2(x - 5) = -4$

50. $-5 + 2(x + 5) = -5$

51. $9x - 2(x + 5) = -10$

52. $-8x - 5(x - 3) = 15$

In Exercises 53-68, solve the equation.

53. $4(-7x + 5) + 8 = 3(-9x - 1) - 2$

54. $-4(-x + 9) + 5 = -(-5x - 4) - 2$

55. $-8(-2x - 6) = 7(5x - 1) - 2$

56. $5(-4x - 8) = -9(-6x + 4) - 4$

57. $2(2x - 9) + 5 = -7(-x - 8)$

58. $-6(-4x - 9) + 4 = -2(-9x - 8)$

59. $6(-3x + 4) - 6 = -8(2x + 2) - 8$

60. $-5(5x - 9) - 3 = -4(2x + 5) - 6$

61. $2(-2x - 3) = 3(-x + 2)$

62. $-2(7x + 1) = -2(3x - 7)$

63. $-5(-9x + 7) + 7 = -(-9x - 8)$

64. $7(-2x - 6) + 1 = 9(-2x + 7)$

65. $5(5x - 2) = 4(8x + 1)$

66. $5(-x - 4) = -(-x + 8)$

67. $-7(9x - 6) = 7(5x + 7) - 7$

68. $-8(2x + 1) = 2(-9x + 8) - 2$

❧ ❧ ❧ Answers ☙ ☙ ☙

1. 1

3. 4

5. -5

7. 0

9. 1

11. -2

13. -5

15. 1

17. 9

19. 7

21. -2

23. 1

25. -3

27. -1

29. 2

31. -2

33. -2

35. 6

37. 50

39. 0

41. -2

43. 7

45. -3

47. 3

49. -1

51. 0

53. 33

55. 3

57. -23

59. 21

61. -12

63. 1

65. -2

67. 0

3.6 Applications

Because we've increased our fundamental ability to simplify algebraic expressions, we're now able to tackle a number of more advanced applications. Before we begin, we remind readers of required steps that must accompany solutions of applications.

Requirements for Word Problem Solutions.

1. **Set up a Variable Dictionary.** You must let your readers know what each variable in your problem represents. This can be accomplished in a number of ways:
   - Statements such as "Let P represent the perimeter of the rectangle."
   - Labeling unknown values with variables in a table.
   - Labeling unknown quantities in a sketch or diagram.
2. **Set up an Equation.** Every solution to a word problem must include a carefully crafted equation that accurately describes the constraints in the problem statement.
3. **Solve the Equation.** You must always solve the equation set up in the previous step.
4. **Answer the Question.** This step is easily overlooked. For example, the problem might ask for Jane's age, but your equation's solution gives the age of Jane's sister Liz. Make sure you answer the original question asked in the problem.
5. **Look Back.** It is important to note that this step does not imply that you should simply check your solution in your equation. After all, it's possible that your equation incorrectly models the problem's situation, so you could have a valid solution to an incorrect equation. The important question is: "Does your answer make sense based on the words in the original problem statement."

Consecutive Integers

The integers are *consecutive*, in the sense that one follows right after another. For example, 5 and 6 are a pair of consecutive integers. The important relation to notice is the fact that the second integer of this pair is one larger than its predecessor. That is, $6 = 5 + 1$.

Consecutive Integers. Let k represent an integer. The next consecutive integer is the integer $k+1$.

Thus, if k is an integer, then $k+1$ is the next integer, $k+2$ is the next integer after that, and so on.

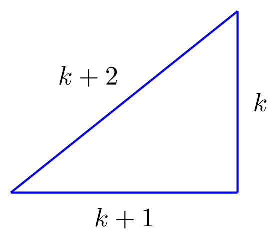
You Try It!

The three sides of a triangle are consecutive integers and the perimeter is 57 centimeters. Find the measure of each side of the triangle.

EXAMPLE 1. The three sides of a triangle are consecutive integers and the perimeter is 72 inches. Find the measure of each side of the triangle.

Solution. We follow the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* In this case, a carefully labeled diagram is the best way to indicate what the unknown variable represents.



In our schematic diagram, we've labeled the three sides of the triangle with expressions representing the consecutive integers k, $k+1$, and $k+2$.

2. *Set up an Equation.* To find the perimeter P of the triangle, sum the three sides.

$$P = k + (k+1) + (k+2)$$

However, we're given the fact that the perimeter is 72 inches. Thus,

$$72 = k + (k+1) + (k+2)$$

3. *Solve the Equation.* On the right, regroup and combine like terms.

$$72 = 3k + 3$$

Now, solve.

$72 - 3 = 3k + 3 - 3$	Subtract 3 from both sides.
$69 = 3k$	Simplify.
$\dfrac{69}{3} = \dfrac{3k}{3}$	Divide both sides by 3.
$23 = k$	Simplify.

4. *Answer the Question.* We've only found one side, but the question asks for the measure of all three sides. However, the remaining two sides can be found by substituting 23 for k into the expressions $k+1$ and $k+2$.

$$k+1 = 23+1 = 24 \qquad \text{and} \qquad k+2 = 23+2 = 25$$

Hence, the three sides measure 23 inches, 24 inches, and 25 inches.

5. *Look Back.* Does our solution make sense? Well, the three sides are certainly consecutive integers, and their sum is 23 inches + 24 inches + 25 inches = 72 inches, which was the given perimeter. Therefore, our solution is correct.

Answer: 18, 19, and 20 cm

□

Consecutive Odd Integers

The integer pair 19 and 21 are an example of a pair of *consecutive odd integers.* The important relation to notice is the fact that the second integer of this pair is two larger than its predecessor. That is, $21 = 19 + 2$.

Consecutive Odd Integers. Let k represent an *odd* integer. The next consecutive odd integer is $k+2$.

Thus, if k is an odd integer, then $k+2$ is the next odd integer, $k+4$ is the next odd integer after that, and so on.

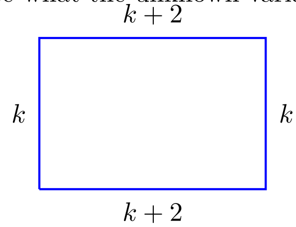
You Try It!

The length and width of a rectangle are consecutive odd integers and the perimeter is 120 meters. Find the length and width of the rectangle.

EXAMPLE 2. The length and width of a rectangle are consecutive odd integers and the perimeter is 168 centimeters. Find the length and width of the rectangle.

Solution. We follow the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* In this case, a carefully labeled diagram is the best way to indicate what the unknown variable represents.



In our schematic diagram, if the width k is an odd integer, then the length $k+2$ is the next consecutive odd integer.

2. *Set up an Equation.* To find the perimeter of the rectangle, sum the four sides.
$$P = k + (k+2) + k + (k+2)$$
However, we're given the fact that the perimeter is 168 centimeters. Thus,
$$168 = k + (k+2) + k + (k+2)$$

3. *Solve the Equation.* On the right, regroup and combine like terms.
$$168 = 4k + 4$$
Now, solve.
$$\begin{aligned} 168 - 4 &= 4k + 4 - 4 && \text{Subtract 4 from both sides.} \\ 164 &= 4k && \text{Simplify.} \\ \frac{164}{4} &= \frac{4k}{4} && \text{Divide both sides by 4.} \\ 41 &= k && \text{Simplify.} \end{aligned}$$

4. *Answer the Question.* We've only found the width, but the question asks for the measure of both the width and the length. However, the length can be found by substituting 41 for k into the expression $k + 2$.
$$\begin{aligned} k + 2 &= 41 + 2 \\ &= 43 \end{aligned}$$
Hence, the width is 41 centimeters and the length is 43 centimeters.

5. *Look Back.* Does our solution make sense? Well, the width is 41 cm and the length is 43 cm, certainly consecutive odd integers. Further, the perimeter would be 41 cm + 43 cm + 41 cm + 43 cm = 168 cm, so our solution is correct.

Answer: $W = 29$ cm, $L = 31$ cm

□

Tables

In the remaining applications in this section, we will strive to show how tables can be used to summarize information, define variables, and construct equations to help solve the application.

You Try It!

EXAMPLE 3. Hue inherits \$10,000 and decides to invest in two different types of accounts, a savings account paying 2% interest, and a certificate of deposit paying 4% interest. He decides to invest \$1,000 more in the certificate of deposit than in savings. Find the amount invested in each account.

Dylan invests a total of \$2,750 in two accounts, a savings account paying 3% interest, and a mutual fund paying 5% interest. He invests \$250 less in the mutual fund than in savings. Find the amount invested in each account.

Solution. We follow the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* We're going to use a table to summarize information and declare variables. In the table that follows, we let S represent the amount Hue invests in the savings account. *Using a variable letter that "sounds like" the quantity it represents is an excellent strategy. Thus, in this case, letting S represent the amount invested in savings is far better than letting x represent the amount invested in savings.*

Account Type	Amount Deposited
Savings Account (2%)	S
Certificate of Deposit (4%)	$S + 1000$
Totals	10000

   Because S represents the investment in savings, and we're told that the investment in the certificate of deposit (CD) is \$1,000 more than the investment in savings, the investment in the CD is therefore $S + 1000$, as indicated in the table.

2. *Set up an Equation.* The second column of the table reveals that the sum of the individual investments in the CD and savings totals \$10,000. Hence, the equation that models this application is

$$(S + 1000) + S = 10000.$$

3. *Solve the Equation.* On the left, regroup and combine like terms.

$$2S + 1000 = 10000$$

   Now, solve.

$$\begin{aligned} 2S + 1000 - 1000 &= 10000 - 1000 && \text{Subtract 1000 from both sides.} \\ 2S &= 9000 && \text{Simplify.} \\ \frac{2S}{2} &= \frac{9000}{2} && \text{Divide both sides by 2.} \\ S &= 4500 && \text{Simplify.} \end{aligned}$$

4. *Answer the Question.* We've only found the investment in savings, but the question also asks for the amount invested in the CD. However, the investment in the CD is easily found by substituting 4500 for S in the expression $S + 1000$.

$$\begin{aligned} S + 1000 &= 4500 + 1000 \\ &= 5500. \end{aligned}$$

   Hence, the investment in savings is \$4,500 and the investment in the CD is \$5,500.

5. *Look Back.* Does our solution make sense? Well, the amount invested in the CD is $5,500, which is certainly $1,000 more than the $4,500 invested in savings. Secondly, the two investments total $\$5,500 + \$4,500 = \$10,000$, so our solution is correct.

Answer: $1,500 in savings, $1,250 in the mutual fund

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You Try It!

David keeps his change in a bowl made by his granddaughter. There is $1.95 in change in the bowl, all in dimes and quarters. There are two fewer quarters than dimes. How many dimes and quarters does he have in the bowl?

EXAMPLE 4. Jose cracks open his piggy bank and finds that he has $3.25 (325 cents), all in nickels and dimes. He has 10 more dimes than nickels. How many dimes and nickels does Jose have?

Solution. We follow the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* We're going to use a table to summarize information and declare variables. In the table that follows, we let N represent the number of nickels from the piggy bank. *Using a variable letter that "sounds like" the quantity it represents is an excellent strategy. Thus, in this case, letting N represent the number of nickels is far better than letting x represent the number of nickels.*

Coins	Number of Coins	Value (cents)
Nickels (5 cents apiece)	N	$5N$
Dimes (10 cents apiece)	$N + 10$	$10(N + 10)$
Totals	—	325

Because there are 10 more dimes than nickels, the number of dimes is $N + 10$, recorded in the second column. In the third column, N nickels, worth 5 cents apiece, have a value of $5N$ cents. Next, $N + 10$ dimes, worth 10 cents apiece, have a value of $10(N + 10)$ cents. The final entry in the column gives the total value of the coins as 325 cents.

2. *Set up an Equation.* The third column of the table reveals that the sum of the coin values is 325 cents. Hence, the equation that models this application is

$$5N + 10(N + 10) = 325,$$

which sums the value of the nickels and the value of the dimes to a total of 325 cents.

3. *Solve the Equation.* On the left, use the distributive property to remove parentheses.

$$5N + 10N + 100 = 325$$

Combine like terms.

$$15N + 100 = 325$$

Now, solve.

$$\begin{aligned} 15N + 100 - 100 &= 325 - 100 && \text{Subtract 100 from both sides.} \\ 15N &= 225 && \text{Simplify.} \\ \frac{15N}{15} &= \frac{225}{15} && \text{Divide both sides by 15.} \\ N &= 15 && \text{Simplify.} \end{aligned}$$

4. *Answer the Question.* We've only found the number of nickels, but the question also asks for the number of dimes. However, the number of dimes is easily found by substituting 15 for N in the expression $N + 10$.

$$\begin{aligned} N + 10 &= 15 + 10 \\ &= 25. \end{aligned}$$

Hence, Jose has 15 nickels and 25 dimes.

5. *Look Back.* Does our solution make sense? Well, the number of dimes is 25, which is certainly 10 more than 15 nickels. Also, the monetary value of 15 nickels is 75 cents and the monetary value of 25 dimes is 250 cents, a total of 325 cents, or \$3.25, so our solution is correct.

Answer: 7 dimes, 5 quarters

□

You Try It!

Emily purchase tickets to the IMAX theater for her family. An adult ticket cost \$12 and a child ticket costs \$4. She buys two more child tickets than adult tickets and the total cost is \$136. How many adult and child tickets did she buy?

EXAMPLE 5. A large children's organization purchases tickets to the circus. The organization has a strict rule that every five children must be accompanied by one adult guardian. Hence, the organization orders five times as many child tickets as it does adult tickets. Child tickets are three dollars and adult tickets are six dollars. If the total cost of tickets is \$4,200, how many child and adult tickets were purchased?

Solution. We follow the *Requirements for Word Problem Solutions.*

1. *Set up a Variable Dictionary.* We're going to use a table to summarize information and declare variables. In the table that follows, we let A represent the number of adult tickets purchased. *Using a variable letter that "sounds like" the quantity it represents is an excellent strategy. Thus, in this case, letting A represent the number of adult tickets is far better than letting x represent the number of adult tickets.*

	Number of Tickets	Cost (dollars)
Adults (\$6 apiece)	A	$6A$
Children (\$3 apiece)	$5A$	$3(5A)$
Totals	—	4200

Because there are 5 times as many children's tickets purchased than adult tickets, the number of children's tickets purchased is $5A$, recorded in the second column. In the third column, $5A$ children's tickets at \$3 apiece will cost $3(5A)$ dollars, and A adult tickets at \$6 apiece will cost $6A$ dollars. The final entry in the column gives the total cost of all tickets as \$4,200.

2. *Set up an Equation.* The third column of the table reveals that the sum of the costs for both children and adult tickets is \$4,200. Hence, the equation that models this application is

$$6A + 3(5A) = 4200$$

which sums the cost of children and adult tickets at \$4,200.

3. *Solve the Equation.* On the left, use the associative property to remove parentheses.

$$6A + 15A = 4200$$

Combine like terms.

$$21A = 4200$$

Now, solve.

$$\frac{21A}{21} = \frac{4200}{21} \qquad \text{Divide both sides by 21.}$$
$$A = 200 \qquad \text{Simplify.}$$

4. *Answer the Question.* We've only found the number of adult tickets, but the question also asks for the number of children's tickets. However, the number of children's tickets is easily found by substituting 200 for A in the expression $5A$.

$$\begin{aligned} 5A &= 5(200) \\ &= 1000. \end{aligned}$$

Hence, 1000 children's tickets and 200 adult tickets were purchased.

5. *Look Back.* Does our solution make sense? Well, the number of children's tickets purchased is 1000, which is certainly 5 times more than the 200 adult tickets purchased. Also, the monetary value of 1000 children's tickets at \$3 apiece is \$3,000, and the monetary value of 200 adult tickets at \$6 apiece is \$1,200, a total cost of \$4,200. Our solution is correct.

Answer: 8 adult and 10 child tickets

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❧ ❧ ❧ Exercises ☙ ☙ ☙

1. The three sides of a triangle are consecutive odd integers. If the perimeter of the triangle is 39 inches, find the lengths of the sides of the triangle.

2. The three sides of a triangle are consecutive odd integers. If the perimeter of the triangle is 51 inches, find the lengths of the sides of the triangle.

3. The width and length of a rectangle are consecutive integers. If the perimeter of the rectangle is 142 inches, find the width and length of the rectangle.

4. The width and length of a rectangle are consecutive integers. If the perimeter of the rectangle is 166 inches, find the width and length of the rectangle.

5. The three sides of a triangle are consecutive even integers. If the perimeter of the triangle is 240 inches, find the lengths of the sides of the triangle.

6. The three sides of a triangle are consecutive even integers. If the perimeter of the triangle is 30 inches, find the lengths of the sides of the triangle.

7. The width and length of a rectangle are consecutive integers. If the perimeter of the rectangle is 374 inches, find the width and length of the rectangle.

8. The width and length of a rectangle are consecutive integers. If the perimeter of the rectangle is 318 inches, find the width and length of the rectangle.

9. The width and length of a rectangle are consecutive odd integers. If the perimeter of the rectangle is 208 inches, find the width and length of the rectangle.

10. The width and length of a rectangle are consecutive odd integers. If the perimeter of the rectangle is 152 inches, find the width and length of the rectangle.

11. The width and length of a rectangle are consecutive even integers. If the perimeter of the rectangle is 76 inches, find the width and length of the rectangle.

12. The width and length of a rectangle are consecutive even integers. If the perimeter of the rectangle is 300 inches, find the width and length of the rectangle.

13. The three sides of a triangle are consecutive even integers. If the perimeter of the triangle is 144 inches, find the lengths of the sides of the triangle.

14. The three sides of a triangle are consecutive even integers. If the perimeter of the triangle is 198 inches, find the lengths of the sides of the triangle.

15. The three sides of a triangle are consecutive integers. If the perimeter of the triangle is 228 inches, find the lengths of the sides of the triangle.

16. The three sides of a triangle are consecutive integers. If the perimeter of the triangle is 216 inches, find the lengths of the sides of the triangle.

17. The width and length of a rectangle are consecutive even integers. If the perimeter of the rectangle is 92 inches, find the width and length of the rectangle.

18. The width and length of a rectangle are consecutive even integers. If the perimeter of the rectangle is 228 inches, find the width and length of the rectangle.

19. The three sides of a triangle are consecutive integers. If the perimeter of the triangle is 105 inches, find the lengths of the sides of the triangle.

20. The three sides of a triangle are consecutive integers. If the perimeter of the triangle is 123 inches, find the lengths of the sides of the triangle.

21. The width and length of a rectangle are consecutive odd integers. If the perimeter of the rectangle is 288 inches, find the width and length of the rectangle.

22. The width and length of a rectangle are consecutive odd integers. If the perimeter of the rectangle is 352 inches, find the width and length of the rectangle.

23. The three sides of a triangle are consecutive odd integers. If the perimeter of the triangle is 165 inches, find the lengths of the sides of the triangle.

24. The three sides of a triangle are consecutive odd integers. If the perimeter of the triangle is 99 inches, find the lengths of the sides of the triangle.

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25. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 8 children must be accompanied by one adult guardian. Hence, the organization orders 8 times as many child tickets as it does adult tickets. Child tickets are \$7 and adult tickets are \$19. If the total cost of tickets is \$975, how many adult tickets were purchased?

26. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 2 children must be accompanied by one adult guardian. Hence, the organization orders 2 times as many child tickets as it does adult tickets. Child tickets are \$6 and adult tickets are \$16. If the total cost of tickets is \$532, how many adult tickets were purchased?

27. Judah cracks open a piggy bank and finds \$3.30 (330 cents), all in nickels and dimes. There are 15 more dimes than nickels. How many nickels does Judah have?

28. Texas cracks open a piggy bank and finds \$4.90 (490 cents), all in nickels and dimes. There are 13 more dimes than nickels. How many nickels does Texas have?

29. Steve cracks open a piggy bank and finds \$4.00 (400 cents), all in nickels and dimes. There are 7 more dimes than nickels. How many nickels does Steve have?

30. Liz cracks open a piggy bank and finds \$4.50 (450 cents), all in nickels and dimes. There are 15 more dimes than nickels. How many nickels does Liz have?

31. Jason inherits \$20,300 and decides to invest in two different types of accounts, a savings account paying 2.5% interest, and a certificate of deposit paying 5% interest. He decides to invest \$7,300 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

32. Trinity inherits \$24,300 and decides to invest in two different types of accounts, a savings account paying 2% interest, and a certificate of deposit paying 5.75% interest. She decides to invest \$8,500 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

33. Gina cracks open a piggy bank and finds \$4.50 (450 cents), all in nickels and dimes. There are 15 more dimes than nickels. How many nickels does Gina have?

34. Dylan cracks open a piggy bank and finds \$4.05 (405 cents), all in nickels and dimes. There are 6 more dimes than nickels. How many nickels does Dylan have?

35. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 2 children must be accompanied by one adult guardian. Hence, the organization orders 2 times as many child tickets as it does adult tickets. Child tickets are \$4 and adult tickets are \$10. If the total cost of tickets is \$216, how many adult tickets were purchased?

36. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 2 children must be accompanied by one adult guardian. Hence, the organization orders 2 times as many child tickets as it does adult tickets. Child tickets are \$7 and adult tickets are \$11. If the total cost of tickets is \$375, how many adult tickets were purchased?

37. Connie cracks open a piggy bank and finds \$3.70 (370 cents), all in nickels and dimes. There are 7 more dimes than nickels. How many nickels does Connie have?

38. Don cracks open a piggy bank and finds \$3.15 (315 cents), all in nickels and dimes. There are 3 more dimes than nickels. How many nickels does Don have?

39. Mary inherits \$22,300 and decides to invest in two different types of accounts, a savings account paying 2% interest, and a certificate of deposit paying 4% interest. She decides to invest \$7,300 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

40. Amber inherits \$26,000 and decides to invest in two different types of accounts, a savings account paying 2.25% interest, and a certificate of deposit paying 4.25% interest. She decides to invest \$6,200 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

41. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 8 children must be accompanied by one adult guardian. Hence, the organization orders 8 times as many child tickets as it does adult tickets. Child tickets are \$6 and adult tickets are \$16. If the total cost of tickets is \$1024, how many adult tickets were purchased?

42. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 3 children must be accompanied by one adult guardian. Hence, the organization orders 3 times as many child tickets as it does adult tickets. Child tickets are \$3 and adult tickets are \$18. If the total cost of tickets is \$351, how many adult tickets were purchased?

43. Alan inherits \$25,600 and decides to invest in two different types of accounts, a savings account paying 3.5% interest, and a certificate of deposit paying 6% interest. He decides to invest \$6,400 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

44. Mercy inherits \$27,100 and decides to invest in two different types of accounts, a savings account paying 3% interest, and a certificate of deposit paying 4% interest. She decides to invest \$8,700 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

45. Tony inherits \$20,600 and decides to invest in two different types of accounts, a savings account paying 2% interest, and a certificate of deposit paying 4% interest. He decides to invest \$9,200 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

46. Connie inherits \$17,100 and decides to invest in two different types of accounts, a savings account paying 2% interest, and a certificate of deposit paying 5.5% interest. She decides to invest \$6,100 more in the certificate of deposit than in savings. Find the amount invested in the savings account.

47. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 2 children must be accompanied by one adult guardian. Hence, the organization orders 2 times as many child tickets as it does adult tickets. Child tickets are \$2 and adult tickets are \$14. If the total cost of tickets is \$234, how many adult tickets were purchased?

48. A large children's organization purchases tickets to the circus. The organization has a strict rule that every 8 children must be accompanied by one adult guardian. Hence, the organization orders 8 times as many child tickets as it does adult tickets. Child tickets are \$8 and adult tickets are \$13. If the total cost of tickets is \$1078, how many adult tickets were purchased?

Answers

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1. 11 in., 13 in., 15 in.

3. 35 in., 36 in.

5. 78 in., 80 in., 82 in.

7. 93 in., 94 in.

9. 51 in., 53 in.

11. 18 in., 20 in.

13. 46 in., 48 in., 50 in.

15. 75 in., 76 in., 77 in.

17. 22 in., 24 in.

19. 34 in., 35 in., 36 in.

21. 71 in., 73 in.

23. 53 in., 55 in., 57 in.

25. 13 adult tickets

27. 12 nickels

29. 22 nickels

31. $\$6,500$

33. 20 nickels

35. 12 child tickets

37. 20 nickels

39. $\$7,500$

41. 16 child tickets

43. $\$9,600$

45. $\$5,700$

47. 13 child tickets

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